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Potential projection operators in the theory of the fractional quantum Hall effect

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Abstract. The projection operators that project the potential energy of electron interactions onto the lowest Landau level of a fractional quantum Hall state are considered for both disk and spherical geometries. The study has been performed by using the Laughlin, the quasi-hole and the quasi-electron wave functions. In the case of spherical geometry, the form of the coordinate-operator transformation obtained in a recent article has been amended. The results obtained are expected to be useful in the calculation of the quasi-hole energy, the quasi-electron energy and the energy gap.

1. Introduction

The fractional quantum Hall effect (FQHE) is one of the most remarkable phenomena discovered in recent years. Considerable effort and much speculation have been centred on investigating such a phenomenon from both the theoretical and experimental points of view (Tsui *et al* [1], Laughlin [2], Jain [3]). The limits of low temperatures and strong magnetic fields tend to restrict the electrons in two dimensions to within the lowest Landau level due to the existence of an energy gap. Girvin and Jach [4] and Girvin [5] studied the Hilbert space of the analytic eigenfunctions of the lowest Landau level in a FQH state. Furthermore, Girvin and Jach [4] developed a simple method for the projection of the quantum operators representing the potential energy of electron interactions onto this space. Their study was performed for the disk geometry, using only the Laughlin wave function. In the present work we have extended their results for harmonic interactions by using the quasi-hole and the quasi-electron wave functions. The results showed that these wave functions are not eigenstates of the harmonic interaction operator, unlike the Laughlin wave function. However, the expressions obtained are useful in calculating expectation values for the quasi-hole energy, the quasi-electron energy and the energy gap.

We have further applied the Girvin and Jach projection approach in the theory of general composite particles and deduced different eigenvalues from those obtained in Asselmeyer and Keiper [6]. We have concluded that the difference is due to the fact that the ground states and wave functions are not identical in the two treatments.

Recently, Alaverdian and Bonesteel [7] have derived an analogous result for the potential projection operator in the case of spherical geometry. However, we found that their result is correct only for projection differential operators of first order. As regards higher orders, we have amended their results and obtained the correct expression for the projection operator. We have further obtained the corresponding projection operators on a sphere for harmonic interactions and for $1/r^2$ and Coulomb potentials. In the case of harmonic interaction, we have

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performed the projection procedure by using the Laughlin and quasi-hole wave functions. The Laughlin wave function was found to be an eigenstate for the harmonic potential and the corresponding eigenvalue has been obtained. For the quasi-hole wave function we have derived an expression which could be used to calculate an expectation value for the quasi-hole energy in spherical geometry.

The present work is arranged in the following way.

In sections 2 and 3 we have dealt with the projection techniques in disk and spherical geometries respectively. The comparison with the recent results of Asselmeyer and Keiper [6] is finally given in section 4.

2. The Hilbert space of disk geometry

In the lowest Landau level the eigenfunctions of the kinetic energy Hamiltonian

$$H_0 = \sum_i \frac{1}{2m_e} \left| \underline{P}_i + \frac{e}{c} \underline{A}_i \right|^2 \tag{1}$$

have the form

$$\Psi[z] = \psi[z] \exp\left[-\frac{1}{4} \sum_i |z_i|^2 \right] \tag{2}$$

where $[z] \equiv (z_1, z_2, \ldots, z_N)$, ψ is a polynomial in the N variables $z_k \equiv x_k - \mathrm{i}y_k$, and N is the number of electrons. Girvin and Jach [4] defined a Hilbert space on the set of the complete analytic functions. The exponential factor in (2) was removed by including it in the measure of the inner product. The inner product was thus defined as

$$(\psi, g) = \int \mathrm{d}\mu[z]\ \psi^*[z] g[z] \tag{3}$$

where

$$\mathrm{d}\mu[z] = \prod_{i=1}^{N} \frac{1}{2\pi} \mathrm{e}^{-|z_i|^2/2}\ \mathrm{d}x_i\ \mathrm{d}y_i. \tag{4}$$

Girvin and Jach [4] have further shown that the projection $\hat{V}$ of the potential operator onto the lowest Landau level can be obtained by ordering each term such that the z^*s all sit to the left and then are replaced by $2\,\partial/\partial z$, i.e. we use the following rules:

$$z^* \longrightarrow 2 \frac{\partial}{\partial z} \tag{5}$$

and

$$V[z^*, z] \longrightarrow \hat{N} V \left[2 \frac{\partial}{\partial z}, z \right] \tag{6}$$

where $\hat{N}$ is the normal ordering operator that keeps all the derivatives to the left.

In the case of harmonic interaction,

$$V[z^*, z] = \frac{1}{2} \lambda^2 \sum_{i<j}^{N} (z_i^* - z_j^*)(z_i - z_j) \tag{7}$$

and accordingly

$$\hat{V} = \lambda^2 \sum_{i<j}^{N} \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j} \right) (z_i - z_j). \tag{8}$$

For the Laughlin wave function,

$$\psi_m[z] = \prod_{k<\ell}^{N} (z_k - z_\ell)^m. \tag{9}$$

Girvin and Jach [4] proved that consequently

$$\hat{V}\psi_m = \lambda^2 N(N-1)\left(1 + \frac{mN}{2}\right)\psi_m. \tag{10}$$

They thus concluded that Laughlin's wave function is an exact eigenfunction of the harmonic interaction. In the present section we extend the work of Girvin and Jach [4] and use the quasi-hole and the quasi-electron wave functions Ψ_m^h, Ψ_m^e. The use of these wave functions is justified since they are eigenfunctions of H_0 with the lowest Landau energy as an eigenvalue. Also, we consider, for simplicity, the case where the quasi-particles are created at the origin. It then follows after removing the exponential term and dropping a factor of 2 from Ψ_m^e that

$$\psi_m^h = \prod_\ell z_\ell \prod_{k<t}^{N} (z_k - z_t)^m \tag{11}$$

and

$$\psi_m^e = \prod_\ell \frac{\partial}{\partial z_\ell} \prod_{k<t} (z_k - z_t)^m. \tag{12}$$

Moreover, for the harmonic interaction (equation (8)) we have

$$\hat{V}\psi = \lambda^2 N(N-1)\psi + \lambda^2 \sum_{i<j} (z_i - z_j)\left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j}\right)\psi \tag{13}$$

where ψ stands for either ψ_m^h or ψ_m^e. It can further be shown that

$$\frac{\partial}{\partial z_i}\psi_m^h = \frac{1}{z_i}\psi_m^h + m\sum_{r\neq i} \frac{1}{z_i - z_r}\psi_m^h \tag{14}$$

and

$$\frac{\partial}{\partial z_i}\psi_m^e = m\sum_{r\neq i}\left[\frac{-2}{(z_i - z_r)^3}\prod_{\ell\neq i,r}\frac{\partial\psi_m}{\partial z_\ell} + \frac{1}{(z_i - z_r)^2}\prod_{\ell\neq r}\frac{\partial\psi_m}{\partial z_\ell} - \frac{1}{(z_i - z_r)^2}\prod_{\ell\neq i}\frac{\partial\psi_m}{\partial z_\ell} + \frac{1}{z_i - z_r}\psi_m^e\right] \tag{15}$$

where ψ_m is the Laughlin wave function and

$$\frac{\partial\psi_m}{\partial z_i} = m\sum_{r\neq i}\frac{1}{z_i - z_r}\psi_m. \tag{16}$$

On substituting (14), (15) in (13) and making use of the relation (Girvin and Jach [4])

$$\sum_{i<j}\sum_{r\neq i,j}\left[\frac{z_i - z_j}{z_i - z_r} - \frac{z_i - z_j}{z_j - z_r}\right] = \frac{N(N-1)(N-2)}{2} \tag{17}$$

we find after some algebra that

$$\hat{V}\psi_m^h = \lambda^2\left[N(N-1)\left(1 + \frac{mN}{2}\right) - I\right]\psi_m^h \tag{18}$$

where

$$I = \sum_{i<j}\frac{(z_i - z_j)^2}{z_i z_j} = \sum_{i<j}\left(\frac{z_i}{z_j} + \frac{z_j}{z_i} - 2\right) = -N(N-1) + \sum_{i<j}\left(\frac{z_i}{z_j} + \frac{z_j}{z_i}\right) \tag{19}$$

and

$$\begin{aligned}\hat{V}\psi_m^e &= \lambda^2 N(N-1)\left(1+\frac{mN}{2}\right)\psi_m^e \\ &+ 2m\lambda^2\sum_{i<j}\left[\frac{-2}{(z_i-z_j)^2}\prod_{\ell\neq i,j}\frac{\partial\psi_m}{\partial z_\ell}+\frac{1}{(z_i-z_j)}\left(\prod_{\ell\neq j}\frac{\partial\psi_m}{\partial z_\ell}-\prod_{\ell\neq i}\frac{\partial\psi_m}{\partial z_\ell}\right)\right] \\ &+ m\lambda^2\sum_{i<j}\sum_{r\neq i,j}(z_i-z_j)\left[-2\left(\frac{1}{(z_i-z_r)^3}\prod_{\ell\neq i,r}\frac{\partial\psi_m}{\partial z_\ell}-\frac{1}{(z_j-z_r)^3}\prod_{\ell\neq j,r}\frac{\partial\psi_m}{\partial z_\ell}\right)\right. \\ &+ \left(\frac{1}{(z_i-z_r)^2}-\frac{1}{(z_j-z_r)^2}\right)\prod_{\ell\neq r}\frac{\partial\psi_m}{\partial z_\ell} \\ &- \left.\left(\frac{1}{(z_i-z_r)^2}\prod_{\ell\neq i}\frac{\partial\psi_m}{\partial z_\ell}-\frac{1}{(z_j-z_r)^2}\prod_{\ell\neq j}\frac{\partial\psi_m}{\partial z_\ell}\right)\right].\end{aligned} \tag{20}$$

The above results show that ψ_m^h, ψ_m^e are not exact eigenfunctions of $\hat{V}$, unlike the ψ_m. However, equations (18), (20) can be used to calculate expectation values of the quasi-hole energy, the quasi-electron energy and the energy gap.

3. The Hilbert space of spherical geometry

The representation of the FQHE on a sphere has the advantage that the wave functions are fully translationally invariant. The first treatment along this direction was given by Haldane [8]. He used the spinor coordinates to determine the position of the electron on the surface of the sphere. Recently, Alaverdian and Bonesteel [7] utilized instead the stereographic coordinates

$$z = x + \mathrm{i}y = \tan\left(\frac{\theta}{2}\right)\exp(-\mathrm{i}\phi) \tag{21}$$

together with the Wu–Yang gauge [9]

$$\underline{A} = \frac{\hbar c S}{eR}\frac{1-\cos\theta}{\sin\theta}\underline{e}_\phi. \tag{22}$$

θ and ϕ are the polar and azimuthal angles, $\underline{e}_\phi$ is a unit vector in the direction perpendicular to the meridian plane, R is the radius of the sphere and $2S$ denotes the number of flux quanta piercing the surface of the sphere. The latter approach has the advantage that the wave functions of the lowest Landau level take a similar form to equation (2) in the case of the disk geometry but with the exponential factor being replaced by $\prod_i(1+|z_i|^2)^{-S}$. We can thus present a Hilbert space with an inner product defined in the same way as in (3). But the measure in this case will be given by

$$\mathrm{d}\mu[z] = \prod_{i=1}^{N}\frac{\mathrm{d}x_i\,\mathrm{d}y_i}{(1+|z_i|^2)^{2S+2}} \tag{23}$$

where a normalization factor has been omitted.

The relation analogous to (5) was also obtained in Alaverdian and Bonesteel [7]. They found that

$$\left(\frac{z^*}{1+|z|^2}\right)^n \longrightarrow \frac{(2S+2-n)!}{(2S+2)!}\frac{\mathrm{d}^n}{\mathrm{d}z^n}. \tag{24}$$

However, we believe that this relation is slightly in error. The correct relation should take the form

$$\left(\frac{z^*}{1+|z|^2}\right)^n \longrightarrow \frac{(2S+1)!}{(2S+n+1)!}\frac{\mathrm{d}^n}{\mathrm{d}z^n}. \tag{25}$$

The proof of (25) follows directly from

$$\left\langle \psi, \frac{\mathrm{d}^n g}{\mathrm{d}z^n} \right\rangle = \frac{(2S+n+1)!}{(2S+1)!} \left\langle \psi, \left(\frac{z^*}{1+|z|^2} \right)^n g \right\rangle. \tag{26}$$

The two transformations (24), (25) agree only for $n = 1$, as they both give

$$\frac{z^*}{1+|z|^2} \longrightarrow \frac{1}{2S+2} \frac{\mathrm{d}}{\mathrm{d}z}. \tag{27}$$

Now, in order to obtain the operator $\hat{V}$ which projects a potential interaction $V[z^*/(1+|z|^2), z]$ onto the lowest Landau level we have to keep all the terms $z^*/(1+|z|^2)$ to the left before using (25). In the rest of this section we proceed to apply this procedure for some specific interactions. To the best of our knowledge this has not been done before. We start from the harmonic interaction for which the potential energy takes the form

$$\begin{aligned} V &= \frac{1}{2}\lambda^2 \sum_{i<j} \underline{r}_{ij} \cdot \underline{r}_{ij} \\ &= 2R^2\lambda^2 \sum_{i<j} \left[\frac{z_i^*}{1+|z_i|^2} \left(1 - \frac{z_j^*}{1+|z_j|^2} z_j \right) - \frac{z_j^*}{1+|z_j|^2} \left(1 - \frac{z_i^*}{1+|z_i|^2} z_i \right) \right] \\ &\quad \times (z_i - z_j). \end{aligned} \tag{28}$$

The second step in (28) was obtained by using the following form for the chord distance between the two points i, j on the surface of the sphere:

$$r_{ij} = \frac{2R|z_i - z_j|}{\sqrt{(1+|z_i|^2)(1+|z_j|^2)}} \tag{29}$$

in addition to the identity

$$\frac{1}{1+|z_i|^2} = 1 - \frac{z_i^*}{1+|z_i|^2} z_i. \tag{30}$$

The application of (27) in (28) yields

$$\hat{V} = \frac{R^2\lambda^2}{S+1} \left[\hat{V}_1 + \frac{1}{2(S+1)} \hat{V}_2 \right] \tag{31}$$

where

$$\begin{aligned} \hat{V}_1 &= \sum_{i<j} \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j} \right) (z_i - z_j) \\ \hat{V}_2 &= \sum_{i<j} \frac{\partial^2}{\partial z_i \, \partial z_j} (z_i - z_j)^2. \end{aligned} \tag{32}$$

The first part $\hat{V}_1$ is identical with the operator obtained in the case of a disk (equation (8)). The projection onto the lowest Landau level can be performed by using the Laughlin wave function. In stereographic coordinates and within the Hilbert space considered, the Laughlin wave function is given by [7]

$$\psi_m[z] = \prod_{k<\ell} (z_k - z_\ell)^m \qquad S = \frac{1}{2} m(N-1). \tag{33}$$

We consequently find from (8) and (10) that

$$\hat{V}_1 \psi_m = N(N-1) \left(1 + \frac{mN}{2} \right) \psi_m. \tag{34}$$

It can further be shown by using (16), (17) together with

$$\frac{\partial^2 \psi_m}{\partial z_j\,\partial z_i} = \frac{m}{(z_i - z_j)^2}\psi_m + m^2\left(\sum_{r\neq i}\frac{1}{z_i - z_r}\right)\left(\sum_{r'\neq j}\frac{1}{z_j - z_{r'}}\right) \tag{35}$$

that

$$\hat{V}_2\psi_m = -\left[(2+3m)\frac{N(N-1)}{2} + mN(N-1)(N-2)\right]\psi_m + m^2\hat{V}_2'\psi_m \tag{36}$$

where

$$\hat{V}_2' = \sum_{i<j}(z_i - z_j)^2\left(\sum_{r\neq i}\frac{1}{z_i - z_r}\right)\left(\sum_{r'\neq j}\frac{1}{z_j - z_{r'}}\right)$$
$$= -N(N-1)\left(N - \frac{3}{2}\right) + \sum_{i<j}\sum_{r\neq i,j}\sum_{r'\neq i,j,r}\frac{(z_i - z_j)^2}{(z_i - z_r)(z_j - z_{r'})}. \tag{37}$$

We have calculated the last term in (37) by utilizing a similar procedure to that used in Girvin and Jach [4] to obtain (17). We have found that

$$\sum_{i<j}\sum_{r\neq i,j}\sum_{r'\neq i,j,r}\frac{(z_i - z_j)^2}{(z_i - z_r)(z_j - z_{r'})} = \frac{-N(N-1)(N-2)(N-3)}{4} \tag{38}$$

and consequently

$$\hat{V}_2' = \frac{-N^2(N-1)^2}{4}. \tag{39}$$

On substituting (39) in (36) and substituting the resulting equation in (31) we obtain

$$\hat{V}\psi_m = \frac{R^2\lambda^2}{S+1}\left\{N(N-1)\left(1 + \frac{mN}{2}\right) - \frac{N(N-1)}{2(S+1)}\left[1 + m\left(N - \frac{1}{2}\right) + \frac{m^2N(N-1)}{4}\right]\right\}\psi_m. \tag{40}$$

But $S = \frac{1}{2}m(N-1)$ and $R = \sqrt{S}$ in units of the magnetic length. It can thus be shown after some manipulation that

$$\hat{V}\psi_m = \frac{\lambda^2 m}{2[m(N-1)+2]}N(N-1)^2\left(1 + \frac{mN}{2}\right)\psi_m. \tag{41}$$

The above result shows that Laughlin's wave function is still an exact eigenfunction of the harmonic interaction on a sphere.

Also, according to Alaverdian and Bonesteel [7] the quasi-hole wave function can be taken in the same form as (11), save that z stands now for the stereographic coordinate. Moreover, $\hat{V}_1\psi_m^h$ is given by (18) with λ^2 being omitted. As regards the operator $\hat{V}_2$, we use (14) in addition to

$$\frac{\partial^2}{\partial z_j\,\partial z_i}\psi_m^h = \left[\frac{1}{z_i z_j} + \frac{m}{(z_i - z_j)^2} + \frac{m}{z_i}\sum_{r\neq j}\frac{1}{z_j - z_r} + \frac{m}{z_j}\sum_{r\neq i}\frac{1}{z_i - z_r} + m^2\left(\sum_{r\neq i}\frac{1}{z_i - z_r}\right)\left(\sum_{r'\neq j}\frac{1}{z_j - z_{r'}}\right)\right]\psi_m^h. \tag{42}$$

It can then be shown by utilizing (17), (38) and after some lengthy but straightforward calculations that

$$\hat{V}_2\psi_m^h = \left\{-N(N-1)\left[1 + m\left(N - \frac{1}{2}\right) + m^2\frac{N(N-1)}{4}\right] + (3+m)I + mJ\right\}\psi_m^h \tag{43}$$

where

$$J = \sum_{i<j} \sum_{r \neq i,j} (z_i - z_j)^2 \left[\frac{1}{z_i(z_j - z_r)} + \frac{1}{z_j(z_i - z_r)} \right]. \tag{44}$$

Consequently, we substitute (18), (43) in (31) and use the two relations $S = \frac{1}{2}m(N-1)$, $R^2 = S$ to finally obtain

$$\hat{V}\psi_m^h = \frac{S\lambda^2}{S+1}\left[\frac{1}{2}N(N-1)\left(1 + \frac{mN}{2}\right) - I\left(1 - \frac{3+m}{2(S+1)}\right) + \frac{m}{2(S+1)}J\right]\psi_m^h. \tag{45}$$

The two terms I, J prevent the quasi-hole wave function ψ_m^h from being an eigenfunction of the projection potential $\hat{V}$. However, equation (45) may be used to calculate an expectation value of the quasi-hole energy.

The above calculations have been performed for the harmonic interaction. As regards an interaction potential of the form $1/r^2$, we have

$$\begin{aligned} V &= \sum_{i<j} \frac{1}{r_{ij}^2} = \frac{1}{4R^2} \sum_{i<j} \frac{(1+|z_i|^2)(1+|z_j|^2)}{(z_i^* - z_j^*)(z_i - z_j)} \\ &= \frac{1}{4R^2} \sum_{i<j} \int_0^\infty \mathrm{d}\lambda \, \exp\left[\frac{-\lambda(z_i^* - z_j^*)(z_i - z_j)}{(1+|z_i|^2)(1+|z_j|^2)}\right]. \end{aligned} \tag{46}$$

The projection operator is, then, given by

$$\hat{V} = \frac{1}{4R^2} \sum_{i<j} \hat{N} \int_0^\infty \mathrm{d}\lambda \, \exp\left[\frac{-\lambda}{2(S+1)}\left(\hat{V}_1 + \frac{\hat{V}_2}{2(S+1)}\right)\right] \tag{47}$$

where $\hat{V}_1$, $\hat{V}_2$ are the two operators given by equation (32) and $\hat{N}$ is the normal ordering operator that keeps all the derivatives to the left.

Similarly, for the Coulomb potential

$$\begin{aligned} V &= \sum_{i<j} \frac{1}{r_{ij}} = \frac{1}{2R} \sum_{i<j} \sqrt{\frac{(1+|z_i|^2)(1+|z_j|^2)}{(z_i^* - z_j^*)(z_i - z_j)}} \\ &= \frac{1}{2R\sqrt{\pi}} \sum_{i<j} \int_0^\infty \mathrm{d}\lambda \, \exp\left[\frac{-\lambda^2(z_i^* - z_j^*)(z_i - z_j)}{(1+|z_i|^2)(1+|z_j|^2)}\right] \end{aligned} \tag{48}$$

and

$$\hat{V} = \frac{1}{2R\sqrt{\pi}} \sum_{i<j} \hat{N} \int_{-\infty}^\infty \mathrm{d}\lambda \, \exp\left[\frac{-\lambda^2}{2(S+1)}\left(\hat{V}_1 + \frac{\hat{V}_2}{2(S+1)}\right)\right]. \tag{49}$$

Results analogous to (47), (49) were obtained in Girvin and Jach [4] for the disk geometry. The difficulty in using these results arises due to the presence of the normal ordering operator, $\hat{N}$.

4. Comparison with other approaches

Recently, Asselmeyer and Keiper [6] investigated the FQHE by considering a model in which the interaction potential $V(\underline{r})$ was assumed to arise due to an average charge e distributed on a homogeneous disk with radius $\ell_r(\underline{B})$ and thickness ℓ_d. The solution of Poisson's equation gives

$$V(\underline{r}) = -\frac{m_e}{2}\omega_0^2 r^2 \tag{50}$$

where

$$\omega_0^2 = \frac{2e^2}{\epsilon m_e \ell_d \ell_r^2(B)} \tag{51}$$

and ϵ is the dielectric constant of the medium. Asselmeyer and Keiper [6] consequently solved the Schrödinger equation of a single electron with the potential energy given by (50). The energy eigenvalues were found to be

$$E'_{nm} = \hbar\tilde{\omega}\left(n + \frac{|m|+1}{2} + \frac{m}{2}\frac{\omega_c}{\tilde{\omega}}\right). \tag{52}$$

Here, the cyclotron frequency $\omega_c = eB/m_e c$, and

$$\tilde{\omega}^2 = \omega_c^2 - 4\omega_0^2 = \frac{e^2 B}{c^2 m_e^2}(B - B_c) \tag{53}$$

where $n = 0, 1, 2, \ldots$ and $m = -\gamma, \ldots, 0, 1, \ldots$. If the interaction potential is neglected entirely, then $\omega_0 = 0$, $\tilde{\omega} = \omega_c$ and the degeneracy of the Landau levels arises from the negative values of m. Thus γ denotes the degree of degeneracy of these levels.

The energy eigenvalues can alternatively be obtained by applying the Girvin and Jach [4] projection approach to a disk. For this purpose we put $V(\underline{r})$ in the form

$$V(\underline{r}) = -\frac{1}{2}\lambda^2 z^* z \qquad \text{where } \lambda^2 = m_e\omega_0^2\ell_0^2. \tag{54}$$

$$\ell_0 = \left(\frac{\hbar}{m_e\omega_c}\right)^{1/2} = \left(\frac{\hbar c}{eB}\right)^{1/2} \tag{55}$$

is the magnetic length and z is measured in units of ℓ_0. The wave function will be taken to be that of Laughlin for a single electron [10]. We thus take

$$\psi_m(z) = z^{|m|} \tag{56}$$

where $|m|$ has been used rather than m, since m takes negative values in Asselmeyer and Keiper [6]. It, consequently, follows by using (5) that

$$\hat{V}\psi_m(z) = -\lambda^2 \frac{\mathrm{d}}{\mathrm{d}z} z^{|m|+1} = -m_e\omega_0^2\ell_0^2(|m|+1)\psi_m = \frac{-\hbar\omega_0^2}{\omega_c}(|m|+1)\psi_m. \tag{57}$$

Accordingly,

$$E_{0,m} = \hbar\omega_c\left[\frac{1}{2} - \frac{\omega_0^2}{\omega_c^2}(|m|+1)\right]. \tag{58}$$

The above result differs from the result of Asselmeyer and Keiper [6] (equation (52)). Moreover, for a numerical comparison we take

$$\frac{\omega_c}{\tilde{\omega}} = \frac{5}{4} \qquad \text{and thus } \frac{\omega_0}{\omega_c} = \frac{3}{10}. \tag{59}$$

Therefore,

$$E_{0,-3} = \frac{7}{40}\hbar\tilde{\omega} \qquad E_{0,-4} = \frac{1}{16}\hbar\tilde{\omega}. \tag{60}$$

The corresponding results of Asselmeyer and Keiper [6] are

$$E'_{0,-3} = \frac{1}{8}\hbar\tilde{\omega} \qquad E'_{0,-4} = 0. \tag{61}$$

The difference between the two sets of results is due to the fact that they stand for two different states represented by two different wave functions. In Asselmeyer and Keiper [6], the wave function is given by

$$\Psi'_{0,m} = z^{|m|} \exp\left[\frac{-|z|^2}{4\ell_0'^2}\right] \qquad \ell_0' = \sqrt{\frac{\hbar}{m_e\tilde{\omega}}} \tag{62}$$

while in Girvin and Jach [4], it is given by the usual lowest-Landau-level wave function:

$$\Psi_{0,m} = z^{|m|} \exp\left[\frac{-|z|^2}{4\ell_0^2}\right] \qquad \ell_0 = \sqrt{\frac{\hbar}{m_e\omega_c}}. \tag{63}$$

5. Conclusions

The extensions considered for the theory of the projection of the potential energy onto the lowest Landau level have yielded important new results for both the disk and spherical geometries, regarding the energy of quasi-holes and quasi-electrons and the energy gap. The results have confirmed that the Laughlin wave function is an exact eigenfunction of the harmonic interaction, unlike the quasi-hole and quasi-electron wave functions. For spherical geometry we have further introduced an exact procedure to obtain the projection operator from the form of the potential energy. The procedure has been applied for the harmonic, for the $1/r^2$ and for the Coulomb potentials.

The results displayed in section 4 showed further that the two approaches of general composite particles and of the projection of the potential term onto the lowest Landau level represent two different ground states.

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